Mixed Anomaly and Global Consistency

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July 6, 2017 @ Goethe Universität Frankfurt

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References: arXiv:1705.01949[hep-th] (JHEP06(2017)102), in preparation

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Motivation: 't Hooft anomaly and Nonperturbative physics

Quantum field theory

Quantum field theory is an important and ubiquitous tool to study low-energy behaviors of quantum many-body systems.

- Nuclear and hadron physics
  $\Rightarrow$ Quantum chromodynamics
- Pion, Kaon physics
  $\Rightarrow$ Nonlinear sigma models
- Quantum Hall effects, Topological states of matters
  $\Rightarrow$ Topological field theories

Symmetry, Topology

Solving QFT is usually very hard.

Example QCD contains quarks and gluons as elementary fields, but it explains physics of hadrons.

Question

Can we say something rigorous even when QFT of our interest is strongly coupled?

Keywords Symmetry and Topology

Especially, we pay attention to anomaly.

't Hooft anomaly

't Hooft anomaly
= Symmetry of the quantum theory that can't be gauged

Example Consider the 4d free massless Dirac fermion

$$S = \int \mathrm{d}^4 x \, \overline{\psi} \gamma_\mu \partial_\mu \psi$$

This quantum theory has the $U(1)_{\mathrm{vector}} \times U(1)_{\mathrm{axial}}$ symmetry. Once one gauges the $U(1)_{\mathrm{vector}}$ symmetry, $U(1)_{\mathrm{axial}}$ is explicitly broken,

$$\partial_\mu j_{5\mu} = \frac{1}{8\pi^2} F_{\mu\nu} \widetilde{F}_{\mu\nu}.$$

One cannot gauge the $U(1)_{\mathrm{vector}} \times U(1)_{\mathrm{axial}}$ symmetry.

Anomaly matching

Theorem

An 't Hooft anomaly is renormalization-group invariant.

Once the 't Hooft anomaly is computed in UV, the same anomaly must be reproduced in IR.

Example

QCD in the chiral limit has the flavor symmetry with an 't Hooft anomaly: $SU(3)_L \times SU(3)_R \times U(1)_V$.
The same anomaly must be reproduced after integrating out all the massive degrees of freedom.
$\Rightarrow$ Chiral symmetry breaking and the WZW term for the pions.

Purpose of this talk

Extend the application of topology and anomaly to study the phase structure of quantum systems

- Explain the technique in quantum mechanical example
- Determine the phase structure of bifundamental gauge theory at nonzero topological angles

(YT, Kikuchi; 1705.01949)

(Related works:
Gaiotto, Kapustin, Komargodski, Seiberg, 1703.00501 about Yang-Mills theory at $\theta=\pi$,
Komargodski, Sharon, Thorngren, Zhou, 1705.04786 about Abelian Higgs model,
Komargodski, Sulejmanpasic, Unsal, 1706.05731 about domain walls in anti-ferromagnet,
Shimizu, Yonekura, 1706.06104 about adjoint QCD.)

Example: Quantum mechanics with the target space $(S^1)^2$

Quantum Mechanical Model

We consider the two-variable QM on a circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$:

$$S_E = \int \mathrm{d}\tau \left[\frac{1}{2}(m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2) + \lambda \cos(q_1 - q_2)\right] - \frac{\mathrm{i}}{2\pi} \int (\theta_1 \mathrm{d}q_1 + \theta_2 \mathrm{d}q_2).$$

Since $\int \mathrm{d}q_i \in 2\pi\mathbb{Z}$,

$$\theta_1 \sim \theta_1 + 2\pi, \quad \theta_2 \sim \theta_2 + 2\pi.$$

Symmetries: $U(1)$ shift symmetry

$$q_1 \mapsto q_1 + \varphi, \ \ q_2 \mapsto q_2 + \varphi.$$

CP ($\mathbb{Z}_2$-reflection) symmetry at $(\theta_1, \theta_2) = (0,0), (\pi, 0), (0, \pi), (\pi, \pi)$

$$q_1 \mapsto -q_1, \ q_2 \mapsto -q_2.$$

Question about the ground state

We want to study the phase structure in the θ_1-θ_2 space.

Question

Where does the phase transition happens?
When does this quantum system have the unique ground state?

At $(\theta_1, \theta_2) = (0, 0)$, there is the unique ground state.

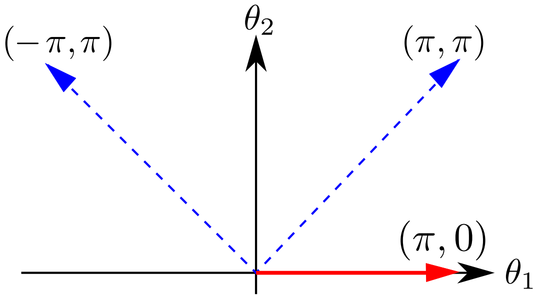
Proof.

At $(\theta_1, \theta_2) = (0, 0)$, the Euclidean path integral has no sign problem.
In quantum mechanics, this means $\exists!$ ground state. $\square$

At nonzero $\theta_{1,2}$, the proof fails, and the uniqueness is nontrivial and not always true.

Search of phase boundaries in the θ_1-θ_2 plane

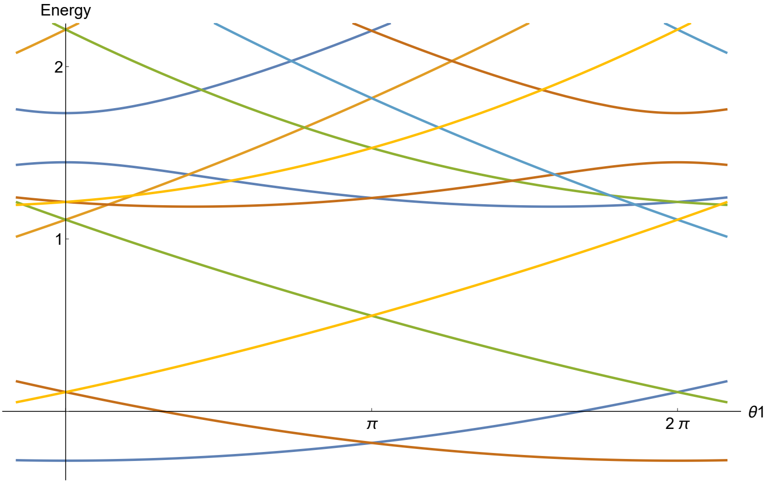
We've seen that $(\theta_1, \theta_2) \simeq (0,0)$ the ground state is unique.



Consider three paths in the θ_1-θ_2 plane:

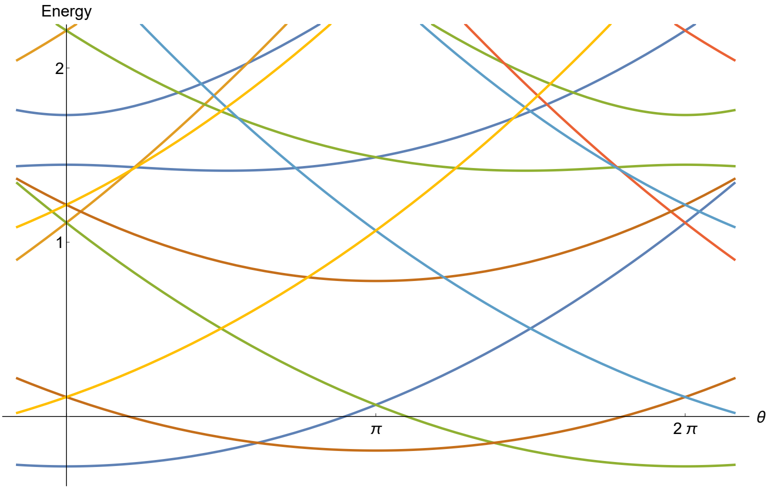
1. $(0,0) \to (\pi, 0)$ 2. $(0,0) \to (\pi, \pi)$ 3. $(0,0) \to (\pi, -\pi)$

Level crossing when $(\theta_1, \theta_2) : (0, 0) \to (\pi, 0)$



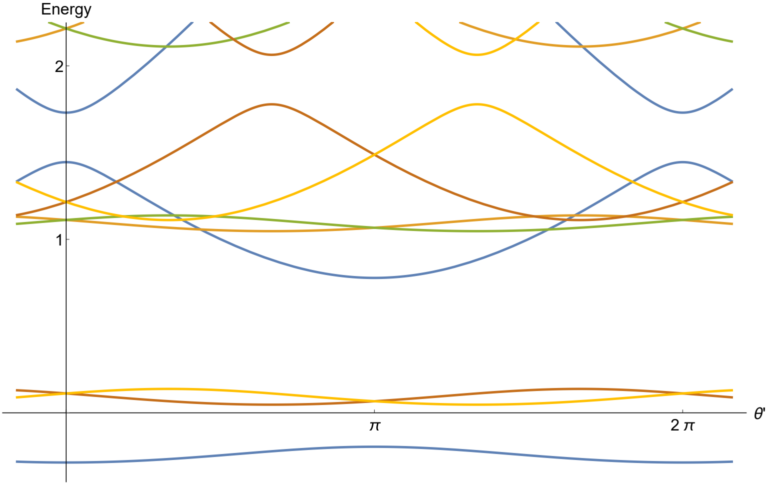
- At $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$, all the states form the pair under CP.

Level crossing when $(\theta_1, \theta_2) : (0,0) \to (\pi, \pi)$



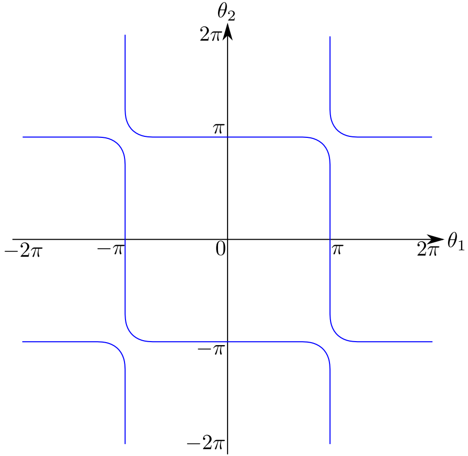
- CP-invariant states at $(0,0)$ break CP at (π,π), and vice versa.

Level crossing when $(\theta_1, \theta_2) : (0, 0) \to (\pi, -\pi)$



- CP-invariant states at $(0, 0)$ is also CP-invariant at $(\pi, -\pi)$.

Phase boundary



Question

Can we obtain this figure without computing energy levels?

Technology: Mixed anomaly and Global consistency

Summary of the results

$$S_E = \int \mathrm{d}\tau \left[\frac{1}{2}(m_1\dot{q}_1^2 + m_2\dot{q}_2^2) + \lambda\cos(q_1 - q_2)\right] - \frac{\mathrm{i}}{2\pi}\int(\theta_1\mathrm{d}q_1 + \theta_2\mathrm{d}q_2).$$

This has $U(1)$ symmetry, and the $(\mathbb{Z}_2)_{CP}$ symmetry also exists at $(\theta_1,\theta_2) = (0,0), (\pi,0), (0,\pi), (\pi,\pi)$.

- At $(\theta_1,\theta_2) = (\pi,0), (0,\pi)$, all the states form the pair under CP.
- CP-invariant states at $(0,0)$ break CP at (π,π), and vice versa.
- CP-invariant states at $(0,0)$ is also CP-invariant at $(\pi,-\pi)$.

Theorem

The above is the consequence of mixed anomaly/global consistency for $U(1)\times(\mathbb{Z}_2)_{CP}$.

Basic idea of the strategy

Since we have symmetries $U(1)$ and $(\mathbb{Z}_2)_{CP}$, we define

$$Z[A,C]=\int \mathcal{D}q \exp\left(-S_E[q,A,C]\right).$$

A: $U(1)$ gauge field, C: $(\mathbb{Z}_2)_{CP}$ gauge field.

We consider the gauge transformation, and find that

$$Z[A+\mathrm{d}\lambda, C+\mathrm{d}\phi]=Z[A,C]\times\exp\left(\mathrm{i}\,\text{“phase”}[A,C,\lambda,\phi]\right).$$

This “phase” functional is RG-invariant ('t Hooft anomaly matching), and thus useful to constrain the low-energy properties.

Gauging the $U(1)$ symmetry

We gauge the $U(1)$ shift symmetry, and call $A = A_0 \mathrm{d}\tau$ the $U(1)$ gauge field.

$$
\begin{aligned}
S_E &= \int_0^\beta \mathrm{d}\tau \left[\frac{1}{2}(m_1(\dot{q}_1 + A_0)^2 + m_2(\dot{q}_2 + A_0)^2) + \lambda \cos(q_1 - q_2)\right] \\
&\quad -\frac{\mathrm{i}}{2\pi}\int (\theta_1(\mathrm{d}q_1 + A) + \theta_2(\mathrm{d}q_2 + A)) \\
&\quad +\mathbf{i}\boldsymbol{p}\int \boldsymbol{A}.
\end{aligned}
$$

The last term is the local contact term (1d Chern-Simons term), and $p \in \mathbb{Z}$.

CP transformation after gauging $U(1)$

Recall that CP flips the sign of $\mathrm{d}q_i \mapsto -\mathrm{d}q_i$, which effectively changes $\theta_i \mapsto -\theta_i$.
Since $\theta_i \sim \theta_i + 2\pi$ before gauging, not only $\theta_i = 0$ but also $\theta_i = \pi$ are CP-invariant candidates.

Meaning of 2π-shift after gauging the symmetry:

$$\begin{aligned} -\frac{\mathrm{i}\pi}{2\pi}\int(\mathrm{d}q_i + A) &\mapsto \frac{\mathrm{i}\pi}{2\pi}\int(\mathrm{d}q_i + A) \\ &= -\frac{\mathrm{i}\pi}{2\pi}\int(\mathrm{d}q_i + A) + \boldsymbol{\mathrm{i}\int(\mathrm{d}q_i + A)}. \end{aligned}$$

Since $\mathrm{i}\int \mathrm{d}q_i \in 2\pi\mathrm{i}\mathbb{Z}$ does not affect the path integral, the last term means that

$$p \mapsto p + 1.$$

Criterion of CP invariance after gauging

Under CP, $\mathrm{i}p\int A$ also flips its sign. The effect of CP is

$$p \mapsto \begin{cases} -p, & (\theta_1,\theta_2)=(0,0), \\ -p+1, & (\theta_1,\theta_2)=(\pi,0),(0,\pi), \\ -p+1+1, & (\theta_1,\theta_2)=(\pi,\pi), \\ -p+1-1, & (\theta_1,\theta_2)=(\pi,-\pi). \end{cases}$$

Therefore, we can gauge the $U(1)$ symmetry without breaking CP if we can choose p s.t.

$$p = \begin{cases} -p, & (\theta_1,\theta_2)=(0,0),(\pi,-\pi), \\ -p+1, & (\theta_1,\theta_2)=(\pi,0),(0,\pi), \\ -p+2, & (\theta_1,\theta_2)=(\pi,\pi). \end{cases}$$

Mixed 't Hooft anomaly at $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$

Try to gauge $U(1)$ at $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$ without breaking CP, then

$$p = -p + 1 \quad \Rightarrow \quad p = \frac{1}{2}.$$

This (half CS term) breaks $U(1)$ gauge invariance under large gauge transformations.

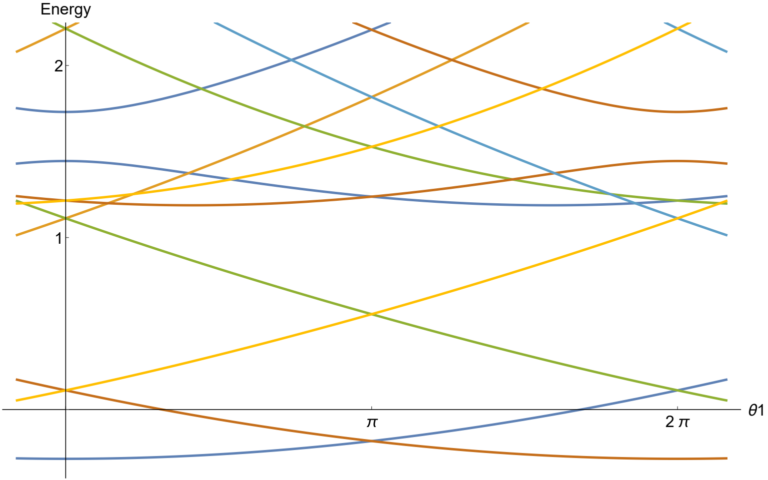
$\Rightarrow U(1)$ and $(\mathbb{Z}_2)_{CP}$ has a mixed anomaly at $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$.

't Hooft anomaly matching claims the following statement:

Theorem

All the states at $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$ must form the pairs under CP.

Level crossing when $(\theta_1, \theta_2) : (0,0) \to (\pi, 0)$



No mixed anomaly at $(\theta_1, \theta_2) = (\pi, \pi)$

Try to gauge $U(1)$ at $(\theta_1, \theta_2) = (\pi, \pi)$ without breaking CP, then

$$p = -p + 2 \quad \Rightarrow \quad p = 1.$$

This is the gauge invariant choice so there is no mixed anomaly.

This means that the CP-invariant states could exist.

Question

Can CP-invariant states at $(\theta_1, \theta_2) = (0, 0)$ be continuously connected to CP-invariant states at $(\theta_1, \theta_2) = (\pi, \pi)$?

The explicit computation suggests No.

- CP-invariant states at $(0, 0)$ break CP at (π, π), and vice versa.

Global inconsistency between $(0,0)$ and (π,π)

If we want to keep CP at $(\theta_1,\theta_2)=(0,0)$ in gauging,

$$p=-p \quad \Rightarrow \quad p=0.$$

If we do the same thing at $(\theta_1,\theta_2)=(\pi,\pi)$ in gauging,

$$p=-p+2 \quad \Rightarrow \quad p=1.$$

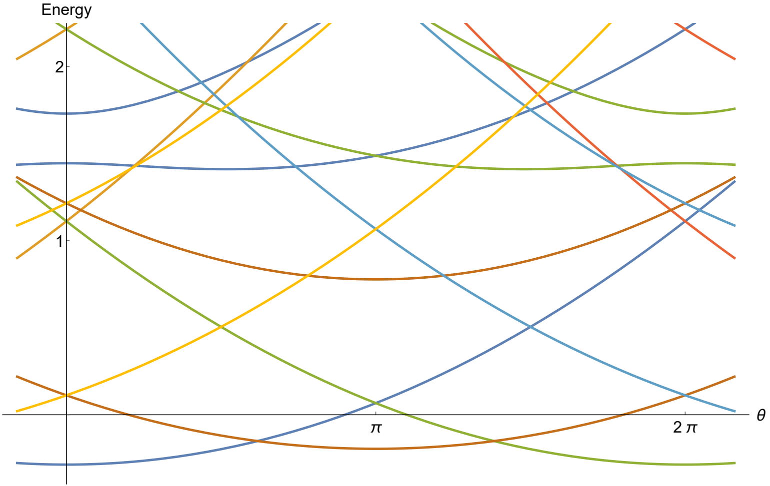
$\Rightarrow$ There is no way to keep the CP symmetry both at $(\theta_1,\theta_2)=(0,0)$ and (π,π) simultaneously.

(Gaiotto, Kapustin, Komargodski, Seiberg; 1703.00501, YT, Kikuchi; 1705.01949)

Theorem

CP-invariant states at $(0,0)$ break CP at (π,π).
CP-invariant states at (π,π) break CP at $(0,0)$.

Level crossing when $(\theta_1, \theta_2) : (0, 0) \to (\pi, \pi)$



Global consistency between $(0,0)$ and $(\pi,-\pi)$

If we want to keep CP at $(\theta_1,\theta_2)=(0,0),(\pi,-\pi)$ in gauging,

$$p=-p \quad \Rightarrow \quad p=0.$$

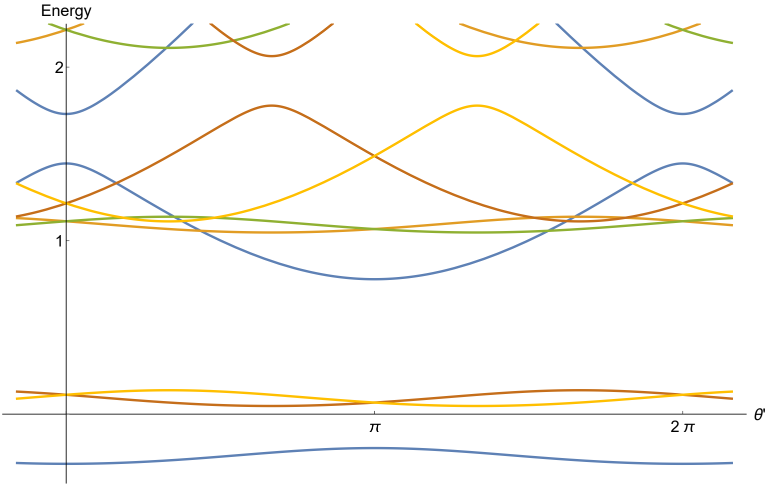
$\Rightarrow$ We can keep the CP symmetry both at $(\theta_1,\theta_2)=(0,0)$ and (π,π) by choosing the same integer p.

Disregarding the accidental degeneracy, we obtain the following:

Theorem

CP-invariant states at $(0,0)$ generically become also CP-invariant at $(\pi,-\pi)$, and vice versa.

Level crossing when $(\theta_1, \theta_2) : (0, 0) \to (\pi, -\pi)$



Brief summary for QM example

By gauging the $U(1)$ symmetry of the full symmetry $U(1) \times (\mathbb{Z}_2)_{CP}$, we can rigorously argue that

- At $(\theta_1, \theta_2) = (\pi, 0), (0, \pi)$, all the states form the pair under CP.
- CP-invariant states at $(0, 0)$ break CP at (π, π), and vice versa.

By assuming "naturalness", we also find

- CP-invariant states at $(0, 0)$ is also CP-invariant at $(\pi, -\pi)$.

Note

For this conclusion, a smaller symmetry $\mathbb{Z}_n \times (\mathbb{Z}_2)_{CP}$ $(< U(1) \times (\mathbb{Z}_2)_{CP})$ is sufficient with some $n \geq 3$.

Consequence

Mixed anomaly and Global consistency put nonperturbative constraints on the spectrum.

Application: Nonperturbative study of bifundamental gauge theories

QCD(BF)

We consider the $SU(n)_1 \times SU(n)_2$ gauge theory,

$$S = \sum_{i=1}^{2} \left\{ -\frac{1}{2g_i^2} \int \mathrm{Tr}(G_i \wedge *G_i) + \frac{\mathrm{i}\theta_i}{8\pi^2} \int \mathrm{Tr}(G_i \wedge G_i) \right\} + \int \mathrm{Tr}\, \overline{\Psi}(\not{D} + m)\Psi,$$

where G_i is the field strength of the $SU(n)_i$ gauge group,

$$G_i = \mathrm{d}a_i + a_i \wedge a_i,$$

and Ψ is the Dirac field in the bifundamental representation,

$$\not{D}\Psi = \gamma^\mu(\partial_\mu \Psi + a_{1\mu}\Psi - \Psi a_{2\mu}).$$

Symmetries

This theory has the symmetry

$$(\mathbb{Z}_n)_{\text{center}} \times (\mathbb{Z}_2)_{CP}.$$

There's the diagonal center $\mathbb{Z}_n$ one-form symmetry, which transforms

$$W_1(C) = \mathrm{Tr}\left[\mathcal{P}\exp \oint_C a_1\right], \quad W_2(C) = \mathrm{Tr}\left[\mathcal{P}\exp \oint_C a_2\right],$$

as ($\omega_n = \exp(2\pi \mathrm{i}/n)$)

$$W_1(C) \mapsto \omega_n W_1(C), \quad W_2(C) \mapsto \omega_n W_2(C).$$

Gauging $(\mathbb{Z}_n)_{\mathrm{center}}$ symmetry

To gauge the $\mathbb{Z}_n$ one-form symmetry, we couple the $SU(n)\times SU(n)$ gauge theory to the $\mathbb{Z}_n$ topological field theory:

$$S_{\mathrm{TFT}} = \frac{\mathrm{i}}{2\pi}\int F\wedge(\mathrm{d}A+nB)+\frac{\mathrm{i}np}{4\pi}\int B\wedge B$$

Here, A is the $U(1)$ (one-form) gauge field, and B is the $U(1)$ (two-form) gauge field. $p\in\mathbb{Z}$ and $p\sim p+n$.

The coupling between a_i and B is given by replacing

$$a_i\Rightarrow \mathcal{A}_i=a_i+\frac{1}{n}A,\quad G_i\Rightarrow \mathcal{G}_i+B.$$

Here, $\mathcal{A}_i$ is the $U(n)=(SU(n)\times U(1))/\mathbb{Z}_n$ gauge potential, and $\mathcal{G}_i=\mathrm{d}\mathcal{A}_i+\mathcal{A}_i\wedge\mathcal{A}_i$.

(YT, Kikuchi; 1705.01949) (cf: Kapustin, Seiberg; 1401.0740, Gaiotto, Kapustin, Seiberg, Willet; 1412.5148)

Comparison with QM and QCD(BF)

	QM	QCD(BF)
Symmetry to be gauged	$U(1)$	$(\mathbb{Z}_n)_{\mathrm{center}}$
Charged observables	$\exp(\mathrm{i}q_i)$	$W_i = \mathrm{tr}[\mathcal{P}\exp\int a_i]$
Gauge field	$A = A_0 \mathrm{d}\tau$	$B = \frac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$
Local counter term	$\mathrm{i}p\int A$	$\frac{\mathrm{i}np}{4\pi}\int B\wedge B$
Minimal coupling	$\mathrm{d}q_i \Rightarrow \mathrm{d}q_i + A$	$G_i \Rightarrow \mathcal{G}_i + B$

Criterion of CP invariance after gauging

Similar computation with the QM example shows that the effect of CP is described by the change of p (mod n)

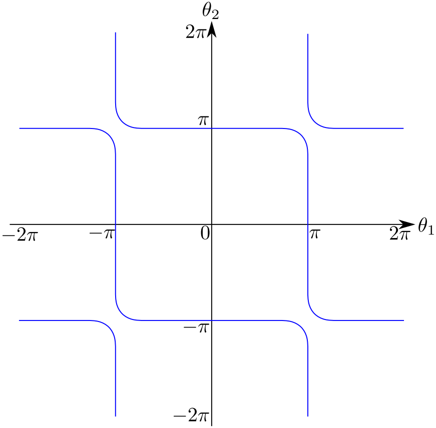
$$p \mapsto \begin{cases} -p, & (\theta_1, \theta_2) = (0,0), (\pi, -\pi), \\ -p+1, & (\theta_1, \theta_2) = (\pi, 0), (0, \pi), \\ -p+1+1, & (\theta_1, \theta_2) = (\pi, \pi). \end{cases}$$

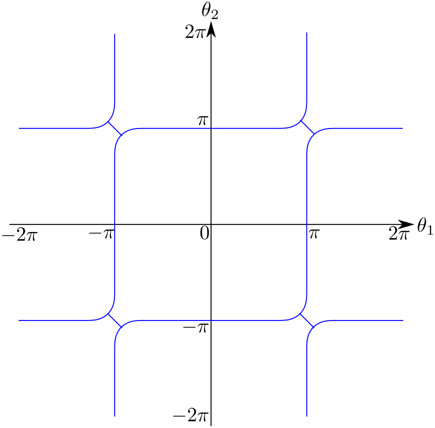
Therefore, we can gauge the $\mathbb{Z}_n$ symmetry without breaking CP if we can choose p (mod n) s.t.

$$p = \begin{cases} -p, & (\theta_1, \theta_2) = (0,0), (\pi, -\pi), \\ -p+1, & (\theta_1, \theta_2) = (\pi, 0), (0, \pi), \\ -p+2, & (\theta_1, \theta_2) = (\pi, \pi). \end{cases}$$

Possible phase boundaries

We get simplest possible phase boundaries ($n \geq 3$):





We assume for this result that the theory is always confining, has the mass gap, and is topologically trivial. (YT, Kikuchi; 1705.01949)

Summary

- Mixed anomaly and global consistency is a powerful nonperturbative tool.
- If there is a mixed anomaly, the states are always degenerate.
- Even if there is no mixed anomaly, the global (in)consistency between two symmetric points restricts the level crossing.
- As one application, we considered the bifundamental gauge theory, and put the nonperturbative constraints on the phase diagram by computing mixed anomaly and global consistency.